HOW MATHEMATICS TEACHERS CAN EXPLAIN MULTIPLICATION AND DIVISION LIKE RENÉ DESCARTES AND ISAAC NEWTON

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If *"the purpose of life is to contribute in some way to making things better"*, how might we make mathematics better?[1] Teachers often explain multiplication and division with repeated addition and subtraction. Yet such approaches do not extend beyond the positive Integers. By contrast, the ideas of René Descartes and Isaac Newton on multiplication and division can be extended from the Naturals to the Reals. So, I reveal how, if they were alive today, they might explain multiplication and division visually in ways seldom seen in western mathematics curriculums.

Background

The 'Cartesian Plane', named after Descartes, has both a horizontal $\boldsymbol{x}$-axis and a vertical $\boldsymbol{y}$-axis, that intersect at zero. The plane thus has four quadrants, the first of which is often used for an area model of multiplication. For example, a rectangle drawn with a base of 8 and a height of 3 will cover 24 'square units' on the Cartesian Plane, as shown as Figure 1. A rectangle with base of 3 and height of 8 also covers 24 square units.

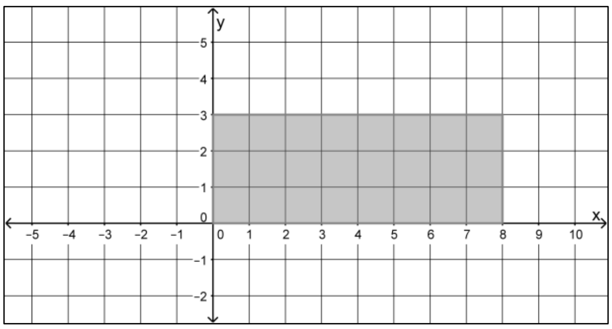

Figure 1. An area model depicting 8 × 3 on the Cartesian Plane.

The 'Cartesian Product', also named after Descartes, consists of a product set formed from two or more other sets. For example, a child has a set of 8 shirts, each a different colour and a set of 3 skirts, each a different colour. Altogether there are 24 different colour combinations of 'shirt and skirt' that can be worn.

1 Quote attributed to American Senator, Robert F. Kennedy.

Modern mathematics began with two great advances from the 1600s. The first was analytic geometry, primarily attributed to Descartes, while the second was calculus, attributed in priority to Isaac Newton and publication to Gottfried Leibniz. However, neither the Cartesian Plane nor the Cartesian Product has anything to do with the original writings of René Descartes on multiplication.

Importantly, Isaac Newton read Descartes' 1637 *La Géométrie* and 1644 *Principia Philosophiae* (Principles of Philosophy). After this, Newton developed calculus and formulated the laws of motion and universal gravitation, later published in his 1687 *Philosophiæ Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy).[2] Having climbed such scientific heights seldom seen before, Newton went on to write *Arithmetica Universalis* (Universal Arithmetic) published in 1707.

So, having built a reputation as a mathematical and scientific genius, perhaps equalled only by Albert Einstein since in impact, did Newton draw upon repeated addition, equal groups, arrays, or area models to explain multiplication? Or, for that matter, did Descartes? No.

Repeated addition and dimension

Today, some might think little about how strange it is that the 'multiplication' of two one-dimensional lines produces two-dimensional area. Yet, if you were to stack an infinite number of horizontal lines 1 metre long side-by-side, (rather than end-on-end), the breadth of that stack of lines would be zero. That is because, as Euclid defined in *Elements* around 300 BCE, *a line is a breadthless length*.[3] So, the repeated addition model, applied to lines, only works in one dimension, when adjoined end-on-end. Similarly, an area has length and breadth only, so if you were to stack an infinite number of areas 1-metre square, the height of that area would also be zero.

Lines cannot be repeatedly added to make area and areas cannot be repeatedly added to make volume, so why pretend they can? This pretence is evident from the widespread use of the area model of multiplication (via the repeated addition of same-size areas) and the *magical* length × width calculation. Similarly, a two-dimensional area multiplied by a perpendicular line gives us 'length × width × height' which *magically* converts the two-dimensional area into a three-dimensional volume. If people have not been confused by this, it is perhaps evident they have not thought about this. Euclid thought about this, which is why he carefully wrote about squares ON a line, and not squares OF a line. The point may be subtle[4], (as well as zero magnitude in all dimensions), yet it is important. Whether you say multiplication is repeated addition or not, an infinite number of points repeatedly added will never make a line, nor lines areas, nor areas volumes.

Descartes' lost logic

If Descartes were alive today, he might be surprised to see so many students being led to believe multiplication is *only* repeated addition. We can explain how 2 × 3 is equal to 2 × (0 + 1 + 1 + 1) and 0 + 2 + 2 + 2. Yet later on with 2 × ⁻3, we get 2 × (0 – 1 – 1 – 1) and 0 – 2 – 2 – 2, so multiplication is repeated subtraction! So, multiplication is much *more* than repeated addition. By adopting the insights of Descartes and Newton, a meaning can be given to multiplication and division as applied to the Real numbers generally. Yet, such ideas are uncommon today, because a London haberdasher, Henry Billingsley, changed Euclid's (proportional) multiplication definition into an illogical

[2] The term 'natural philosophy' evolved into physical sciences and physics.

[3] Euclid's *Elements*, Book I, Definition 2.

[4] A point has zero magnitude, or as Euclid wrote in *Elements*, Book I, Definition 1, 'a point is that which has no part'.

repeated addition algorithm (Crabtree, Dec. 2016).

The first heading in Descartes' *La Géométrie* was *Problems the construction of which require only straight lines and circles*. Descartes' first diagram depicted the multiplication of line segments via similar triangles. The diagram wasn't new, as it was taken from Euclid's *Elements*.[5] Euclid defined a number as a multitude of units, and thus, for Euclid, the unit was not a number. The innovation of Descartes, almost 2000 years later, was to make one of the three given straight lines a unit (with length 1) while the other two straight lines were the two lines (numbers) to be multiplied. Translated from the French, we read:

> ...in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication)...

Descartes' original multiplication diagram and explanation is shown as Figure 2.

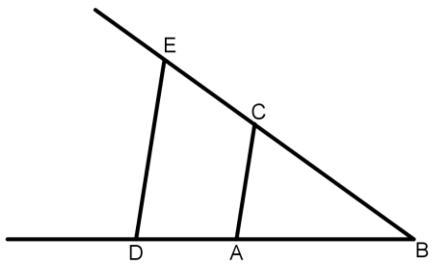

Figure 2. The diagram Descartes used to explain multiplication.

> For example, let AB be taken as unity, and let it be required to multiply BD by BC, then I have only to join the points A and C, and draw DE parallel to CA; and BE is the product of this Multiplication.

We can update Descartes' diagram for our 'Cartesian Plane' because the angle at B in the triangle is irrelevant, and also works as a right angle as shown in Figure 3.

[5] The diagram Descartes tweaked was from Euclid's *Elements*, Book VI, Definition 12, *To find a fourth proportional to three given straight lines*.

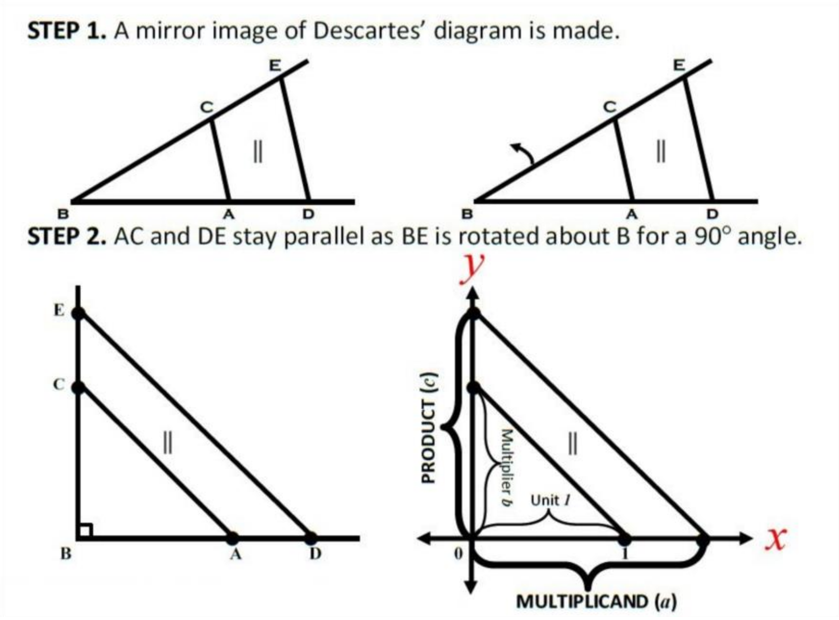

*Figure 3. How to create similar triangles to reveal **a** multiplied by **b** = **c***

Thinking outside the square

Given Descartes depicted multiplication with triangles, we first test whether or not an area model can emerge, not from unit squares, but from unit triangles. The short answer is yes, as shown in Figure 4.

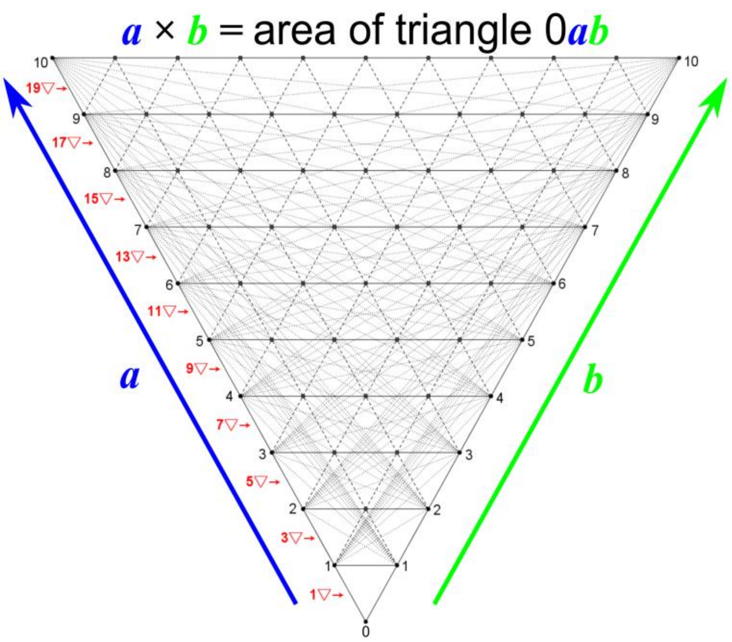

Figure 4. An area model for triangular units. Interactive applet at www.jonathancrabtree.com/mathematics/the-multiplication-triangle

The above 10 × 10 multiplication table contains 100 triangles just as the standard 10 × 10 table contains 100 squares. With, for example, 5 × 5, the triangle contained by the points 0, 5 and 5 contains 25 triangular units. (Our standard table with products in squares is better pedagogically, as areas in the 'real world' are quoted in square units.)

From Descartes to Newton

In 1707, Newton followed Descartes' with a similar explanation of multiplication, translated from the Latin in *Arithmetica Universalis*, shown below as Figure 5.

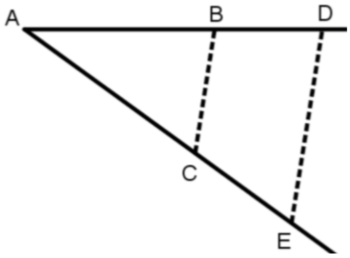

Figure 5. The diagram Newton used to explain multiplication.

> If you were to multiply any two Lines, AC and AD, by one another, take AB for Unity, and draw BC, and parallel to it DE, and AE will be the product of this multiplication, because it [AE] is to AD as AC, [is] to AB Unity.

Area model vs. DesCartesian model

Because multiplication is a proportional concept, in all arithmetical equations, as the *Unit* is to the *Multiplier*, the *Multiplicand* is to the *Product*. With the simple example of two multiplied by three, written 2 × 3, as 1 varies to make 3, so 2 varies to make 6. Such proportional covariation (PCV) failed to emerge, either via the area model of multiplication or repeated addition model (Crabtree, 2016). As a line segment of 1 is to a line segment of 3, a line segment of two must be to a line segment of 6. To say a line of 1 is to a line of 3 as a line of 2 is to a rectangle of 6 is nonsense. Yet, with the *DesCartesian Multiplication Model*, we have a multiplication model that preserves proportional relationships, as is evident in Figure 6.

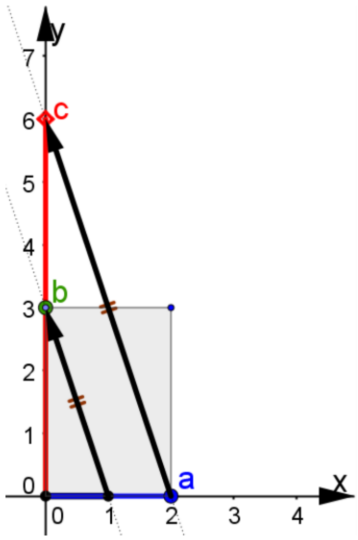

*Figure 6. The standard area model for 2 × 3 alongside Descartes' proportional approach, which reveals 'as 1 is to 3, so 2 is to 6'. To multiply 2 by 3, a line is drawn from the Unit **1** to the Multiplier **b**, which is 3. Then a second line, parallel to the first drawn, is drawn from the Multiplicand **a**, to produce the Product **c**.*

We can imagine 'two square units stacked three times' in the above, and also demonstrate commutativity of multiplication as shown in Figure 7, where we might imagine 'three square units stacked two times'.

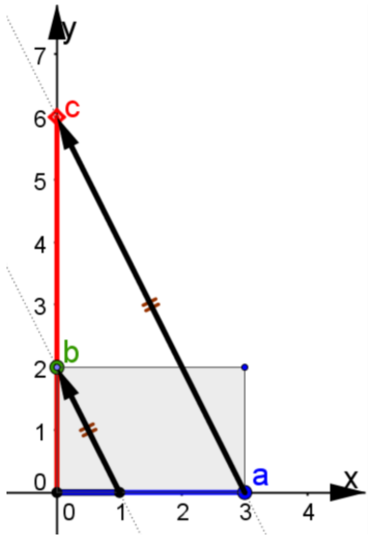

Figure 7. The standard area model for 3 × 2 alongside Descartes' proportional approach, which via similar triangles, reveals 'as 1 is to 2, so 3 is to 6'.

From the diagrams of Descartes and Newton, we have straight lines going up and down (albeit at an angle) and horizontal lines going left and right. Importantly (for what we are about to develop) with the following comments, Newton introduced the notion of positive and negative line segments that encompassed irrationals.

> In Geometry, if a line drawn any certain way be reckon'd for affirmative, then a line drawn the contrary way may be taken for negative: As if AB be drawn to the right; and BC to the left; and AB be reckon'd affirmative, then BC will be negative...
>
> *and*
>
> Multiplication is also made use of in Fractions and Surds, to find a new Quantity in the same Ratio (whatever it be) to the Multiplicand, as the Multiplier has to Unity.

By the 1680s Newton had drawn curves in all four quadrants, consistent with our understanding of the Cartesian Plane. These were published in 1704 as an appendix to *Opticks*. However, with the mathematics community focussed on what was to become algebraic geometry, the DesCartesian Multiplication Model, as applicable to the Reals, appears to have been overlooked. Thus, primary mathematics teachers focus on the first quadrant of the Cartesian Plane for the simple reason an area cannot be 'less than zero'. Yet, such difficulties dissolve with the *DesCartesian Multiplication Model*. For example, beyond the first quadrant, the combinations of ±2 × ±3 are shown in Figure 8.

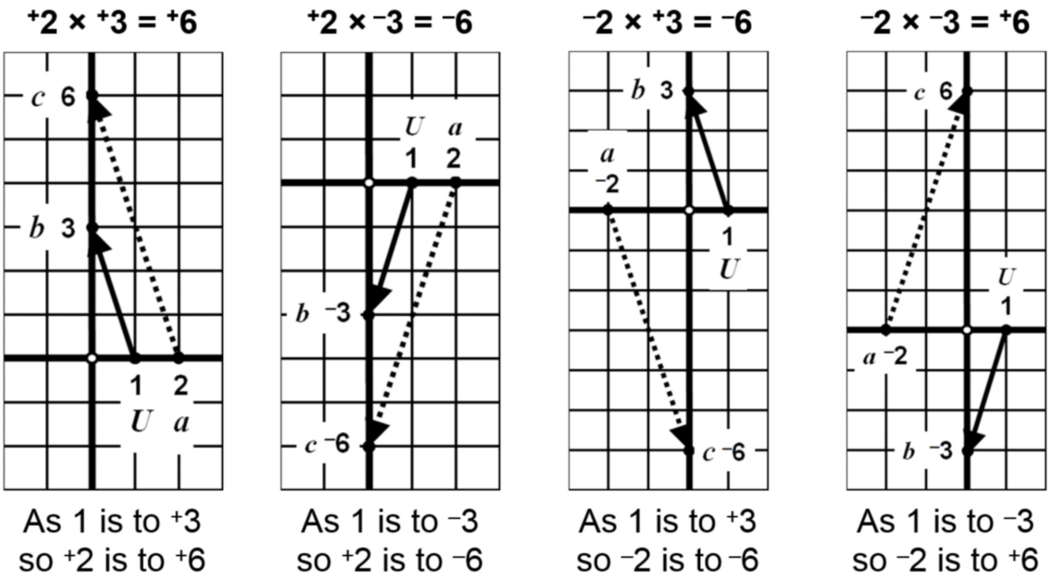

*Figure 8. Combinations of ±2 × ±3 depicted with 'DesCartesian Multiplication', where **1** is the Unit, **b** is the Multiplier, **a** is the Multiplicand and **c** is the Product. Interactive applet at* www.jonathancrabtree.com/mathematics/what-is-descartesian-multiplication

Regardless of the sign, in all cases 'as ***1*** is to ***b***, so ***a*** is to ***c***'. What we also see, is how the multiplication of similarly signed factors results in a positive product, while differently signed factors result in a negative product.

DesCartesian division

After Descartes explained multiplication, he used the same diagram (Figure 1.) to explain division. "If it be required to divide BE by BD, I join E and D, and draw AC parallel to DE; then BC is the result of the division".

Thus, for division all we need do is 'invert' the multiplier, so instead of it encapsulating the ratio ***1*** to ***b***, it becomes the divisor encapsulating the ratio ***b*** to ***1***. In multiplication, whatever we did to the Unit to make the Multiplier, we do to the Multiplicand to make the Product. Unsurprisingly, (given division is the inverse operation of multiplication), in division whatever we did to the Divisor to make the Unit, we do to the Dividend to make the Quotient. As usual, a picture is worth a thousand words, so Figure 9 depicts 9 ÷ 3. Again, consistent with proportional covariation (PCV) however 3 is varied to make 1, so 9 is varied to make the quotient.

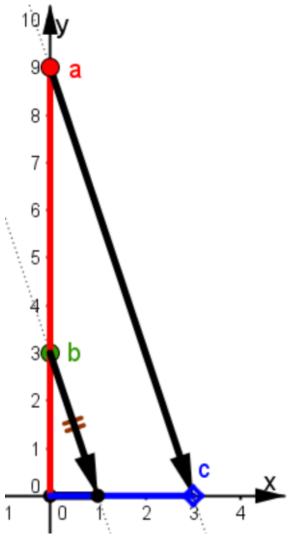

*Figure 9. 'DesCartesian Division' depicting 9 ÷ 3. To divide 9 by 3, a line is drawn from the Divisor **b**, to the Unit **1**. Then a second line, parallel to the first drawn, is drawn from the Dividend **a**, to produce the Quotient **c**.*

The DesCartesian approach simplifies the 'impossible'

Division has both a 'repeated subtraction' (quotitive) model and an 'equal shares' (partitive) model. Yet, without citing sign laws, if mathematics teachers rely only on these models, they simply CANNOT explain how to solve $^{+}9 \div {}^{-}3$. There are no negative threes in positive nine and you cannot divide nine into negative three groups. However, with the *DesCartesian Division Model*, there is little difficulty, as shown below in Figure 10.

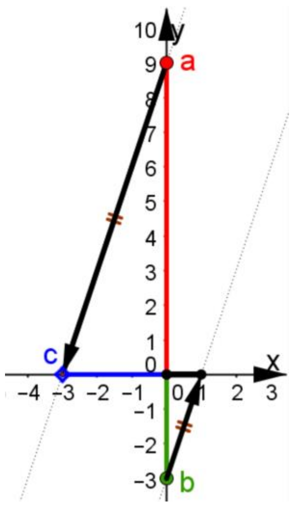

Figure 10. 'DesCartesian Division' depicting 9 ÷ ⁻3.

The DesCartesian diagram depicting $^{-}9 \div {}^{-}3$ is shown in Figure 11. Put simply, as $^{-}3$ is to 1, so $^{-}9$ is to 3. In accordance with the laws of sign, our negative dividend divided by a negative divisor produces a positive quotient. To vary $^{-}3$ and make 1, we take one of three equal parts of $^{-}3$, which is $^{-}1$, and change its sign to make 1. Having done that, we take one of three equal parts of $^{-}9$, to get $^{-}3$ and change its sign to make 3.

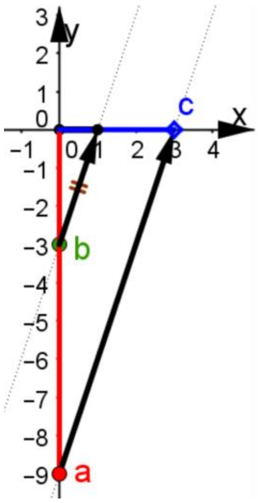

Figure 11. 'DesCartesian Division' depicting ⁻9 ÷ ⁻3. Interactive applet at
www.jonathancrabtree.com/mathematics/what-is-descartesian-division

Final thoughts

Because we can make triangles between 0 and any two other points, one on each axis, the DesCartesian model for multiplication and division applies to the set of Real numbers. These approaches to multiplication and division are in fact, applications of proportional covariation (PCV). From this long overdue historical evolution of arithmetical ideas, with further research and development by the mathematics education community, together, we might implement new approaches and unlock more useful and powerful ideas for teaching mathematics. Thank you for reading.

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